

HOMOLOGY OF GL_3 OF FUNCTION RINGS OF ELLIPTIC CURVES

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ABSTRACT. The note provides a description of the homology of GL_3 over function rings of affine elliptic curves over arbitrary fields, following the earlier work of Takahashi and Knudson in the case GL_2 . Some prospects for applications to K-theory of elliptic curves are also discussed.

1. INTRODUCTION

This note is a step towards understanding the homology groups $H_\bullet(GL_n(k[C]), \mathbb{Z})$ of general linear groups over function rings of affine elliptic curves (with one point O at infinity). At present, only the case $(P)GL_2$ is sufficiently understood, due to the work of Takahashi [Tak93] and Knudson [Knu99a], see in particular [Knu01, Section 4.5]. Takahashi computed the action of $GL_2(k[C])$ on the Bruhat-Tits tree associated to $K = k(C)$ with the valuation corresponding to the point O at infinity and found a fundamental domain. Knudson then worked out the equivariant spectral sequence and computed $H_\bullet(PGL_2(k[C]), \mathbb{Z})$.

This note takes the next step, working out both the equivariant cell structure of the $GL_3(k[C])$ -action on the building and evaluating the resulting equivariant spectral sequence. This can be done completely explicitly, via matrix calculations as in [Tak93] (and this actually is the way I first hit on the results in the fall of 2013). However, in the meantime there is a more conceptual way of formulating (and proving) the results which should also be applicable to higher rank computations.

Let me outline the results of Knudson and Takahashi from this more conceptual perspective. The quotient $GL_2(k[C]) \backslash \mathfrak{T}_C$ of the Bruhat-Tits tree is related to rank two vector bundles on the complete elliptic curve \overline{C} . The quotient is contractible, and can be retracted onto (the star of) the central point o of Takahashi's fundamental domain which corresponds to the unique stable bundle $E(2, 1)$ on \overline{C} which restricts trivially to C . After removing this central point, the fundamental domain decomposes into disjoint trees indexed by $\mathbb{P}^1(k)$ corresponding to semistable rank two bundles $\mathcal{L} \oplus \mathcal{L}^{-1}$ with trivial determinant on \overline{C} . On each of these trees, the homology of the automorphism groups of the vector bundles is constant (using homotopy invariance), and we can rewrite Knudson's formula [Knu01, Theorem 4.5.2] as

$$H_i(PGL_2(k[C]), \mathbb{Z}) \cong \bigoplus_{E \in \mathcal{M}_{2,o}(\overline{C})} H_i(\text{Aut}(E)/k^\times, \mathbb{Z}).$$

The results for $GL_3(k[C])$ follow the same path: the quotient $GL_3(k[C]) \backslash \mathfrak{X}_C$ is described in terms of vector bundles; the structure of the subcomplex of bundles with non-trivial automorphisms can be understood in terms of suitable moduli spaces of vector bundles; and the remaining part of the quotient is controlled by the stable bundles.

There are two important points where the results for GL_3 differ from those for GL_2 . First, the subcomplex of unstable bundles is essentially zero-dimensional for GL_2 , but is essentially 1-dimensional in the case GL_3 . Second, the quotient $\mathrm{GL}_2(k[C]) \backslash \mathfrak{T}_C$ is contractible because there is a unique stable bundle $E(2, 1)$, but the quotient $\mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C$ is not contractible because there are two stable bundles $E(3, 1)$ and $E(3, 2)$.

Acknowledgement: The motivation for this work goes back to early 2011 when Guido Kings asked me if Knudson's work on K-theory of elliptic curves using buildings could be pushed further. It took me a while to realize what interesting ramifications this question had, and it will take me a while longer to work out the answer. The present note is a snapshot of work in progress...

2. BUILDINGS AND VECTOR BUNDLES

In the following, let k be a field, \overline{C} be an elliptic curve with k -rational point O , and set $C = \overline{C} \setminus \{O\}$. The function field $K = k(C)$ has a valuation $v = v_O$, and \mathfrak{X}_C will denote the Bruhat-Tits building associated to GL_3 and the valuation v . Recall that the building \mathfrak{X}_C is a contractible simplicial complex whose set of 0-simplices is the set $\mathrm{GL}_3(K)/(K^\times \cdot \mathrm{GL}_3(\mathcal{O}_v))$ of equivalence classes of lattices, and whose higher simplices are given by chains of lattice inclusions.

The group $\mathrm{GL}_3(k[C])$ acts on the building \mathfrak{X}_C , and the 0-simplices of the quotient $\mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C$ can be identified with the set of rank three vector bundles on \overline{C} which restrict trivially to C modulo tensoring with the ideal sheaf \mathcal{I}_O . This identification is well-known, in the case of GL_2 it can be found in [Ser80]. Under this identification, the stabilizer of the 0-simplex corresponding to the vector bundle E on \overline{C} is the automorphism group $\mathrm{Aut}(E)$.

Recall Atiyah's classification [Ati57] of vector bundles on elliptic curves over algebraically closed fields, which was extended to arbitrary fields by Pumplün [Pum04]. There is a unique indecomposable bundle $E(r, d)$ of rank r with determinant $\mathcal{I}_O^{\otimes d}$ (and all indecomposable bundles can be obtained this way by replacing \mathcal{I}_O by arbitrary line bundles \mathcal{L}). These bundles are stable if $(r, d) = 1$, and semi-stable otherwise. Moreover, the Harder-Narasimhan filtration for vector bundles over elliptic curves splits, so that each vector bundle is S-equivalent to a direct sum of indecomposable bundles as above.

Finally, recall from [Tu93] that the moduli space $\mathcal{M}_{r,d}(\overline{C})$ of S-equivalence classes of semistable vector bundles of rank r and degree d on \overline{C} is isomorphic to $\mathrm{Sym}^h \overline{C}$ with $h = (r, d)$. Under this isomorphism, the determinant map $\det : \mathcal{M}_{r,d}(\overline{C}) \rightarrow \mathrm{Jac}_d(\overline{C})$ is identified with the Abel-Jacobi map, so that for a fixed line bundle \mathcal{L} , the moduli space $\mathcal{M}_{r,\mathcal{L}}$ of semistable rank r vector bundles with determinant \mathcal{L} is identified with \mathbb{P}^{h-1} .

3. UNSTABLE BUNDLES AND THE HECKE GRAPH

Definition 3.1. Define \mathfrak{P}_C to be the subcomplex of the building \mathfrak{X}_C consisting of cells whose stabilizer in $\mathrm{GL}_3(k[C])$ contains a non-central subgroup. This is called the parabolic subcomplex, and its $\mathrm{GL}_3(k[C])$ -equivariant homology

$$\hat{H}_\bullet(\mathrm{GL}_3(k[C]), M) := H_\bullet^{\mathrm{GL}_3(k[C])}(\mathfrak{P}_C, M)$$

is called parabolic homology.

By a result of Atiyah, a vector bundle E on \overline{C} is stable if and only if $\mathrm{Aut}(E) \cong k^\times$ consists of scalars. Therefore, a 0-simplex is in $\mathrm{GL}_3(k[C]) \backslash \mathfrak{P}_C$ if and only if it corresponds to a bundle which is not stable, i.e., is different from $E(3,1)$ and $E(3,2)$.

The complex $\mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C$ can be described in terms of certain moduli spaces of vector bundles. For this, let me define a graph called “Hecke graph”, the name chosen because the graph is related to Hecke operators connecting moduli spaces of vector bundles.

Definition 3.2. *Let C be an affine elliptic curve in Weierstrass normal form. Define the coloured Hecke graph $\Gamma(C, k)$ to be*

$$\mathcal{M}_{2,1}(\overline{C})(k) \xleftarrow{\alpha} \mathcal{M}_{2,0}(\overline{C})(k) \xrightarrow{\beta} \mathcal{M}_{3,0}(\overline{C})(k).$$

The map $\alpha : \mathrm{Sym}^2(\overline{C}) \rightarrow \mathrm{Jac}(\overline{C})$ is the determinant, and the map $\beta : \mathrm{Sym}^2(\overline{C}) \rightarrow \mathbb{P}^2$ is the symmetric square of the (given choice of) covering $\overline{C} \rightarrow \mathbb{P}^1$ branched at four points.

The vertices in $\mathcal{M}_{2,1}(\overline{C}) \cong \mathrm{Jac}(\overline{C})$ correspond to rank three bundles of the form $E = E_{\mathcal{L}}(2,1) \oplus \mathcal{L}^{-1}$, and we have $\mathrm{Aut}(E) \cong k^\times \times k^\times$. The vertices in $\mathcal{M}_{3,0}(\overline{C}) \cong \mathbb{P}^2$ correspond to rank three bundles over \overline{C} which geometrically split as direct sums of three line bundles of degree 0. There are five types of such bundles, depending on the structure of the scheme-theoretic fiber of β over $x \in \mathbb{P}^2$:

- Split bundles $E = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ with pairwise non-isomorphic summands, in which case $\mathrm{Aut}(E) \cong (k^\times)^3$.
- Split bundles $E = \mathcal{L}_1 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$ with $\mathcal{L}_1 \not\cong \mathcal{L}_2$, in which case $\mathrm{Aut}(E) \cong \mathrm{GL}_2(k) \times k^\times$.
- Split bundles $E = \mathcal{L} \oplus \mathcal{L} \oplus \mathcal{L}$ in which case $\mathrm{Aut}(E) \cong \mathrm{GL}_3(k)$.
- Partially split bundles $E = \mathcal{A} \oplus \det \mathcal{A}^{-1}$ with \mathcal{A} an indecomposable geometrically split bundle corresponding to a degree 2 point $a \in \mathrm{Jac}(\overline{C})$ with residue field $k(a)$. In this case, $\mathrm{Aut}(E) \cong k(a)^\times \times k^\times$.
- Indecomposable but geometrically split bundles $E = \mathcal{B}$ corresponding to a degree 3 point $b \in \mathrm{Jac}(\overline{C})$ with residue field $k(b)$. In this case, $\mathrm{Aut}(E) \cong k(b)^\times$.

The set $\mathcal{M}_{2,0}(\overline{C}) \cong \mathrm{Sym}^2(\overline{C})$ of edges corresponds to bundles of the form $E = E_{\mathcal{L}}(2,0) \oplus \mathcal{L}^{-1}$. Up to a unipotent subgroup, $\mathrm{Aut}(E) \cong k^\times \times k^\times$.

Theorem 3.3. *(1) After a suitable subdivision of $\Gamma(C, k)$, the above assignment gives rise to a simplicial weak equivalence*

$$\mu : \Gamma(C, k) \rightarrow \mathrm{GL}_3(k[C]) \backslash \mathfrak{P}_C.$$

- (2) The assignment $\mathrm{Aut} : \Gamma(C, k) \mapsto \mathcal{G}r$ turns $\Gamma(C, k)$ into a graph of groups. If k is infinite or $\mathrm{char} k$ is invertible in the coefficients, then the map μ induces isomorphisms in Borel-equivariant homology of the developments of the respective complexes of groups.*

The proof of the theorem is a long list of tedious linear algebra computations: Atiyah’s classification provides a list of all bundles E on \overline{C} which are not stable, and for each of these, one has to determine the action of $\mathrm{Aut}(E)$ on $\mathrm{Lk}_{\mathfrak{X}_C}(e)$, where e is a lift of the point of $\mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C$ corresponding to E . Eventually, this is the study of the action of subgroups of GL_3 on projective homogeneous

varieties GL_3/P - tedious, but nothing conceptually deep. A byproduct of the computations are explicit representing cocycles for all rank three bundles on \overline{C} with trivial restriction to C .

Theorem 3.4. *Denote by $\mathcal{E} = \mathcal{M}_{2,0}(\overline{C})(k)$ the set of edges of $\Gamma(C, k)$, and by $\mathcal{V} = \mathcal{M}_{2,1}(\overline{C})(k) \cup \mathcal{M}_{3,0}(\overline{C})(k)$ the set of vertices of $\Gamma(C, k)$. There is a long exact sequence computing parabolic homology:*

$$\cdots \rightarrow \bigoplus_{E \in \mathcal{E}} \mathrm{Aut}(E) \xrightarrow{\alpha_* + \beta_*} \bigoplus_{V \in \mathcal{V}} \mathrm{Aut}(V) \rightarrow \widehat{H}_\bullet(\mathrm{GL}_3(k[C]), \mathbb{Z}) \rightarrow \cdots$$

where α_* and β_* denote the induced maps on homology for the inclusions of edge groups into vertex groups.

This follows from Theorem 3.3 by applying the spectral sequence for $\mathrm{GL}_3(k[C])$ -equivariant homology of \mathcal{P}_C . The identification with the Hecke graph of groups shows that the spectral sequence reduces to a long exact sequence exhibiting parabolic homology as cone of the inclusion of edge groups into vertex groups.

Corollary 3.5. *Replacing $\mathrm{Aut}(E)$ by $\mathrm{Aut}(E)/k^\times$ in Theorem 3.4 provides a long exact sequence computing $H_\bullet(\mathrm{PGL}_3(k[C]), \mathbb{Z})$ for $\bullet \geq 3$.*

This follows because the building for $\mathrm{GL}_3(k[C])$ has dimension 2, and all the cells in $\mathrm{PGL}_3(k[C]) \backslash \mathfrak{X}_C$ which are not in the parabolic subcomplex have trivial stabilizer.

It is then fairly straightforward to compute Hilbert-Poincaré series for the homology $H_\bullet(\mathrm{PGL}_3(\mathbb{F}_q[C]), \mathbb{F}_\ell)$ with $\ell \nmid q$.

4. STABLE BUNDLES AND THE STEINBERG MODULE

It remains to describe the part of $\mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C$ which is not contained in the parabolic subcomplex. As noted earlier, the only 0-simplices not in $\mathrm{GL}_3(k[C]) \backslash \mathfrak{P}_C$ are the points x_i , $i = 1, 2$ corresponding to the stable bundles $E(3, i)$. Denote by $\mathfrak{S}_C = \mathrm{Star}_{\mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C}(x_1) \cup \mathrm{Star}_{\mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C}(x_2)$ the union of the stars of x_i . Note that the boundary $\mathrm{Lk}_{\mathfrak{X}_C}(x_i)$ of these stars can be identified as spherical building for $\mathrm{GL}_3(k)$.

Theorem 4.1. *The complex \mathfrak{S}_C is a double cone over the spherical building for $\mathrm{GL}_3(k)$. The inclusion $\mathfrak{S}_C \subseteq \mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C$ is a weak equivalence of simplicial sets, hence $\mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C$ has the homotopy type of a suspension of the spherical building for $\mathrm{GL}_3(k)$.*

Theorem 4.2. *(1) The inclusion of the parabolic subcomplex gives rise to an exact sequence of complexes of $\mathrm{GL}_3(k[C])$ -modules*

$$0 \rightarrow C_\bullet(\mathfrak{P}_C) \rightarrow C_\bullet(\mathfrak{X}_C) \rightarrow C_\bullet(\mathfrak{X}_C/\mathfrak{P}_C) \rightarrow 0.$$

(2) The quotient $\mathfrak{Q}_C = \mathrm{GL}_3(k[C]) \backslash \mathfrak{X}_C/\mathfrak{P}_C$ has the homotopy type of a wedge $\mathfrak{S}_C \vee \Sigma \Gamma(C, k)$, and

$$\begin{aligned} & H_\bullet^{\mathrm{GL}_3(k[C])}(\mathfrak{X}_C/\mathfrak{P}_C, \mathbb{Z}) \\ & \cong H_\bullet(\mathfrak{Q}_C, \mathbb{Z}) \otimes H_\bullet(k^\times, \mathbb{Z}) \\ & \cong H_\bullet(k^\times, \mathbb{Z}) \oplus (H_{\bullet-2}(k^\times, \mathbb{Z}) \otimes (\mathrm{St}_3(k) \oplus H_1(\Gamma(C, k), \mathbb{Z}))) \end{aligned}$$

where $\mathrm{St}_3(k)$ is the Steinberg module for $\mathrm{GL}_3(k)$.

(3) *The long exact homology sequence associated to the sequence in 1) takes the form*

$$\begin{aligned} \cdots \rightarrow \widehat{H}_\bullet(GL_3(k[C]), \mathbb{Z}) \rightarrow H_\bullet(GL_3(k[C]), \mathbb{Z}) \rightarrow \\ \rightarrow H_\bullet(k^\times, \mathbb{Z}) \oplus (H_{\bullet-2}(k^\times, \mathbb{Z}) \otimes (St_3(k) \oplus H_1(\Gamma(C, k), \mathbb{Z}))) \rightarrow \cdots \end{aligned}$$

The theorem follows directly from the description of the quotient in Theorems 3.3 and 4.1. The map $H_\bullet(GL_3(k[C]), \mathbb{Z}) \rightarrow H_\bullet(k^\times, \mathbb{Z})$ is the one induced from the determinant. Moreover, it is easy to see that the restriction of the boundary map to

$$(H_{\bullet-2}(k^\times, \mathbb{Z}) \otimes H_1(\Gamma(C, k), \mathbb{Z})) \rightarrow \widehat{H}_{\bullet-1}(GL_3(k[C]), \mathbb{Z})$$

is injective. However, at the moment I do not have a formula for the differential

$$(H_{\bullet-2}(k^\times, \mathbb{Z}) \otimes St_3(k)) \rightarrow \widehat{H}_{\bullet-1}(GL_3(k[C]), \mathbb{Z}).$$

For $PGL_3(k[C])$, there is a similar formula. It is in fact easier, as the $PGL_3(k[C])$ -equivariant homology of the quotient $\mathfrak{X}_C/\mathfrak{P}_C$ reduces to $St_3(k) \oplus H_1(\Gamma(C, k), \mathbb{Z})$ in degree 2 and is trivial in degrees ≥ 3 .

Corollary 4.3. *Let $k = \mathbb{F}_q$. Then $\dim_{\mathbb{Q}} H_2(GL_3(k[C]), \mathbb{Q}) = q^3$.*

By the results of Harder [Har77], we know that H_0 and H_2 are the only possibly non-trivial rational homology groups in this case. However, the explicit dimension computation seems to be new. The above result implies the existence of a huge number of “non-detectable” cohomology classes in $H^2(GL_3(k[C]), \mathbb{Z})$, i.e., classes which restrict trivially to the diagonal matrices or any finite subgroups of $GL_3(k[C])$. This is yet another case where the function field analogue of Quillen’s conjecture on cohomology of S -arithmetic groups fails.

5. ALGEBRAIC K-THEORY OF ELLIPTIC CURVES

Of course, this research was started in the hope that it might have applications to algebraic K-theory of elliptic curves. Although at present, there is no explicit K-theoretic consequence of the above computations, looking at the building may provide an alternative approach to the construction of the motivic weight two complex for elliptic curves: part (3) of Theorem 4.2 gives rise to a map

$$\sigma : St_3(k) \rightarrow \left(\widehat{H}_1(GL_3(k[C]), \mathbb{Z}) / H_1(\Gamma, \mathbb{Z}) \right).$$

The cokernel of σ is identified with $\ker(\det : H_1(GL_3(k[C])) \rightarrow H_1(k^\times))$, and stabilization results for the homology of linear groups tell us that $H_1(GL_3(k[C])) \cong H_1(GL_\infty(k[C]))$, and hence

$$\ker(\det : H_1(GL_3(k[C])) \rightarrow H_1(k^\times)) \cong SK_1(k[C]).$$

In particular, an explicit computation of σ (which at present is not yet available) would provide an alternative presentation of $SK_1(k[C])$. This might shed light on Vaserstein’s conjecture which predicts $SK_1(k[C])$ to be torsion if k is a number field (or equivalently, predicts $SK_1(\overline{\mathbb{Q}}[C]) = 0$).

The kernel of σ is identified with $H_2(GL_3(k[C]))/\widehat{H}_2(GL_3(k[C]))$, and stabilization results tell us that the natural map $H_2(GL_3(k[C])) \rightarrow H_2(GL_\infty(k[C])) \cong K_2(k[C])$ is surjective. Again, this provides an alternative way of constructing elements in $K_2(k[C])$.

Putting these things together, we see that there is a (yet unspecified) subgroup $\mathcal{R}_C \subseteq \ker \sigma \subseteq \mathrm{St}_3(k)$ (very likely depending on the curve C) such that the following sequence is exact

$$0 \rightarrow K_2^{\{2\}}(k[C]) \rightarrow \mathrm{St}_3(k)/\mathcal{R}_C \xrightarrow{\sigma} \left(\widehat{H}_1(\mathrm{GL}_3(k[C]), \mathbb{Z}) / H_1(\Gamma, \mathbb{Z}) \right) \rightarrow \mathrm{SK}_1(k[C]) \rightarrow 0$$

where

$$K_2^{\{2\}}(k[C]) = H_2(\mathrm{GL}_\infty(k[E])) / \mathrm{Im}(\widehat{H}_2(\mathrm{GL}_3(k[C])))$$

denotes the “rank two” quotient of $K_2(k[C])$. By stabilization, $H_2(\mathrm{GL}_4(k[C])) \cong K_2(k[C])$, hence the set \mathcal{R} of relations could be determined by similar investigations for the action of $\mathrm{GL}_4(k[C])$ on the associated building.

I would expect a strong relation between the complex $[\sigma]$ and the motivic weight two complex of [GL98, Theorem 1.5]. Recall that the theorem of Goncharov and Levin states that the following is (after tensoring with \mathbb{Q}) an exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{Tor}(k^\times, \mathrm{Jac}(\overline{C})) \rightarrow H^0(\overline{C}, \mathcal{K}_2) / K_2(k) \rightarrow \\ \rightarrow B_3^*(\overline{C}) \rightarrow k^\times \otimes \mathrm{Jac}(\overline{C}) \rightarrow \ker(H^1(\overline{C}, \mathcal{K}_2) \rightarrow k^\times) \rightarrow 0 \end{aligned}$$

so that $\tau : B_3^*(\overline{C})_{\mathbb{Q}} \rightarrow k^\times \otimes \mathrm{Jac}(\overline{C})_{\mathbb{Q}}$ is a motivic weight two complex for the elliptic curve \overline{C} . See [GL98] for notation and definitions of elliptic Bloch groups.

There is an obvious conjecture relating the complex of Goncharov-Levin (which is related to the complete curve \overline{C}) to the complex $[\sigma]$ described in this note (which is related to the open curve C):

Conjecture 1. *I expect that there is a morphism of complexes*

$$\begin{array}{ccc} B_3^*(\overline{C})_{\mathbb{Q}} & \xrightarrow{\quad\quad\quad} & (\mathrm{St}_3(k)/\mathcal{R}_C)_{\mathbb{Q}} \\ \tau \downarrow & & \downarrow \sigma \\ (k^\times \otimes \mathrm{Jac}(\overline{C}))_{\mathbb{Q}} & \longrightarrow & \left(\widehat{H}_1(\mathrm{GL}_3(k[C]), \mathbb{Q}) / H_1(\Gamma, \mathbb{Q}) \right) \end{array}$$

which is natural in k and C , such that the induced maps on homology

$$K_2^{[2]}(\overline{C})_{\mathbb{Q}} \rightarrow K_2^{\{2\}}(C)_{\mathbb{Q}} \quad \text{and} \quad K_1^{[2]}(\overline{C})_{\mathbb{Q}} \rightarrow \mathrm{SK}_1(C)_{\mathbb{Q}}$$

agree with the natural restriction maps associated to $C \hookrightarrow \overline{C}$.

If true, the conjecture would imply that the complex $[\sigma]$ is an integral refinement of the motivic weight two complex.

For the map $k^\times \otimes \mathrm{Jac}(\overline{C}) \rightarrow \left(\widehat{H}_1(\mathrm{GL}_3(k[C]), \mathbb{Q}) / H_1(\Gamma, \mathbb{Q}) \right)$, an element $u \otimes \mathcal{L}$ naturally determines an element $u \in \mathrm{Aut}(\mathcal{L} \oplus \mathcal{L}^{-1})^{\mathrm{ab}}$ (scaling by u on the first summand), and the image under the composition $\mathrm{Aut}(\mathcal{L} \oplus \mathcal{L}^{-1})^{\mathrm{ab}} \hookrightarrow H_1(\mathrm{GL}_2(k[C])) \rightarrow H_1(\mathrm{GL}_3(k[C]))$ lies in $\widehat{H}_1(\mathrm{GL}_3(k[C]))$ and is independent of all choices. For the map $B_3^*(\overline{C})_{\mathbb{Q}} \rightarrow (\mathrm{St}_3(k)/\mathcal{R}_C)_{\mathbb{Q}}$, a divisor $D \in \mathbb{Z}[\overline{C}(k)]$ should be mapped to some linear combination of 2-simplices corresponding to elementary transformations on bundles $\mathcal{V} \oplus \det \mathcal{V}^{-1}$ with $\det \mathcal{V}^{-1}$ concentrated in D . The crucial thing missing before the above can be made precise is the explicit description of the map σ and the relations \mathcal{R}_C .

As a final remark, one should note the structural similarity between the above construction and the motivic weight two complex for fields (due to Bloch-Wigner, Suslin, Dupont-Sah, etc):

$$0 \rightarrow \widetilde{\mathrm{Tor}}(\mu_k, \mu_k) \rightarrow K_3^{\mathrm{ind}}(k) \rightarrow \mathcal{P}(k) \rightarrow \Lambda^2(k^\times) \rightarrow K_2(k) \rightarrow 0.$$

The complex itself arises from a differential in the spectral sequence computing the $GL_2(k)$ -equivariant homology of the Čech resolution of $\mathbb{P}^1(k)$; the part $\Lambda^2(k^\times)$ arises from the homology of stabilizer subgroups, and the scissors congruence part $\mathcal{P}(k)$ is a quotient of $\mathbb{Z}[\mathbb{P}^1] \cong St_2(k)$ modulo the dilogarithmic functional equations. All this looks very similar to what's done for elliptic curves in the present note.

The motivic weight two complexes (both for fields as well as for elliptic curves) exhibit a relation between buildings for GL_n and cycle descriptions for algebraic K-theory which is not yet sufficiently understood.

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